QUASIPROJECTIVE VARIETIES ADMITTING ZARISKI DENSE ENTIRE HOLOMORPHIC CURVES

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Abstract. Let X be a complex quasiprojective variety. A result of Noguchi-Winkelmann-Yamanoi shows that if X admits a Zariski dense entire curve, then its quasi-Albanese map is a fiber space. We show that the orbifold structure induced by a properly birationally equivalent map on the base is special in this case. As a consequence, if X is of log-general type with $\bar{q}(X) \geq \dim X$, then any entire curve is contained in a proper subvariety in X.

1. Introduction and the statement of the main result

This paper deals with a question of Campana concerning the characterization of complex algebraic varieties that admits Zariski dense entire holomorphic curves. This problem for an algebraic surface not of log-general type nor a very general algebraic K3 surface was solved completely by [2, 3]. Campana in [5] introduced the notion of special varieties which is a practical way to extend the so far conjectural characterization to higher dimensions. Using a recent result of Noguchi-Winkelmann-Yamaoi [21], we verify one direction of this characterization here for all algebraic varieties whose quasi-Albanese map is generically finite.

Given a complex projective manifold $\bar{X}$ with a normal crossing divisor on it, we call the pair $X = (\bar{X}, D)$ a log-manifold. Recall that there is a locally free subsheaf of the holomorphic tangent sheaf of $\bar{X}$, called the log-tangent sheaf of X, which we denote by $\bar{T}_X$. It is the sheaf of holomorphic vector fields leaving D invariant. Its dual $\bar{\Omega}_X = \bar{T}_X^{\vee}$ is called the log-cotangent sheaf of X and $\bar{K}_X = \det \bar{\Omega}_X$ the log-canonical sheaf of X. Their sections are called logarithmic 1-forms, respectively logarithmic volume forms. Here, and later we will consistently abuse notation and identify holomorphic vector bundles with their sheaves of sections. We will abuse the notation further at times and identify a line bundle with a divisor it corresponds to, for example in the identification $\bar{K}_X = K_{\bar{X}}(D) = K_{\bar{X}} + D$. We first give some proper birational invariants of $\bar{X} \setminus D$, which we will also identify with X by a standard abuse of notation.

DEFINITION 1.1. *With this setup, we define the log-irregularity of X by*

$$\bar{q}(X) = \dim H^0(\bar{\Omega}_X)$$

and we define the log-Kodaira dimension of X by $\bar{\kappa}(X) = \kappa(\bar{K}_X)$, where the Kodaira dimension for an invertible sheaf L is given by

$$\kappa(L) = \limsup_{m\to\infty} \frac{\log \dim H^0(L^{\otimes m})}{\log m}.$$

We also define, see [5, 16], *the essential or the core dimension of X by*

$$\kappa'_+(X) = \max\{\ p \mid L \hookrightarrow \bar{\Omega}^p_X \ \ \text{is an invertible subsheaf with } \kappa(L) = p,\ 0 \leq p \leq \dim X\ \}$$

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It is an easy fact that the Kodaira dimension of an invertible sheaf L is invariant under positive tensor powers of L and so the Kodaira dimension κ makes sense for $\mathbb{Q}$-invertible sheaves of the form $L(A)$ where A is a $\mathbb{Q}$-divisor. We recall also the fact that $\kappa(L) \in \{\infty, 0, \ldots, \dim(X)\}$ for a $\mathbb{Q}$-invertible sheaf L and that sections of powers of L, if exist, define a dominant rational map I_L to a projective variety of dimension $\kappa(L)$, called the Iitaka fibration of L. We usually allow I_L to be defined on any smooth birational model of $\bar{X}$ and choose a model on which I_L is a morphism. With such a choice, recall that the general (in fact generic) fibers F of I_L are connected and $\kappa(I|_F) = 0$, see e.g. [24].

Let X_0 be a quasiprojective variety and $\bar{X}_0$ a projectivization. We recall that a log-resolution of X_0 is a birational morphism $r : \bar{X} \to \bar{X}_0$ where $(\bar{X}, D)$ is a log-manifold with $D = r^{-1}(\bar{X}_0 \backslash X_0)$. Such a resolution exists by the resolution of singularity theorem. The Hartog extension theorem allows us to define $\bar{q}$, $\bar{\kappa}$ and κ'_+ for X_0 by taking them to be those of a log-resolution. These are thus proper birational invariants of X_0. Here, a proper birational map between two quasi-projective varieties are just compositions of proper birational morphisms and their inverses. Another proper birational invariant is given by the (quasi-)Albanese map of X_0, which is an algebraic morphism

$$\alpha_X : X \to \mathrm{Alb}(X) =: \mathrm{Alb}(X_0)$$

defined for any log-resolution $(\bar{X}, D)$ by line integrals of logarithmic 1-forms on $\bar{X} \setminus D$ with a choice of base point outside D where $\mathrm{Alb}(X)$ is a $\bar{q}(X_0)$ dimensional semi-abelian variety called the Albanese variety of X, see for example [20]. Implicit here is the invariance of $\mathrm{Alb}(X)$ and the compatibility of the Albanese map among log-resolutions. We recall that given a compactification of $\mathrm{Alb}(X_0)$, there exist a log-resolution of X_0 (a compactification of X_0 by normal crossing divisors in the case X_0 is smooth) over which α_X extends to a morphism. We recall also that a semi-abelian variety is a complex abelian group T that admits a semidirect product structure via a holomorphic exact sequence of groups

$$0 \to (\mathbb{C}^*)^k \to T \xrightarrow{\pi} A \to 0\ , \tag{1.1}$$

where A is an abelian variety and $k \geq 0$. It follows that the algebro-geometric image of the Albanese map (or the Albanese image) of X_0, its dimension as well as the Albanese variety are proper birational invariants of X_0.

DEFINITION 1.2. *We say that X_0 is* ***special*** *if $\kappa'_+(X_0) = 0$ and that it is of* ***general type*** *(or if more precision is required, of* ***log-general type****) if $\bar{\kappa}(X_0) = \dim X_0$.*

Recall that a holomorphic image of a curve in X_0 is called algebraically degenerate if it is not Zariski dense. Our main theorems in this paper is as follows.

THEOREM 1.3. *Let X be a complex quasi-projective variety with $\bar{q}(X) \geq \dim X$. Then every entire holomorphic curve in X is algebraically degenerate if X is not special. Alternatively X admits a Zariski dense entire holomorphic curve only if X is special.*

COROLLARY 1.4. *With the same hypothesis on X, let $f : \mathbb{C} \to X$ be holomorphic and nontrivial. If X is of general type, then there is a proper subvariety of X containing $f(\mathbb{C})$.*

We note that [22] has proved the same theorem with κ'_+ replaced by $\bar{\kappa}$ but with the additional hypothesis that the Albanese map of X is proper and generically finite. However, without the properness condition for the Albanese map, the birational condition of $\bar{\kappa}(X) = 0$

is not implied by the condition that X admits a Zariski dense holomorphic image of $\mathbb{C}$, see [7].

An important part of this paper is an adaptation to the context of special varieties of the results of Noguchi-Winkelmann-Yamanoi [20, 21, 22] concerning varieties that admit finite maps to semi-Abelian varieties. All the results on special varieties used for the main theorem here are worked out here from scratch independently of previous sources. The second author has spoken about the result on surfaces at a workshop at the Fields Institute in 2008 that claimed the connection with the characterization by special varieties. This connection, at least in one direction of the characterization, is fully worked out here for all dimensions. The first author would like to thank Gerd Dethloff for valuable discussions on Nevanlinna theory and especially for the last part of the proof in proposition 4.5 of the paper. He would also like to thank Frédéric Campana for agreeing on certain new terminologies introduced in this paper, especially the use of "base-special" to characterize a notion introduced and the accompanying use of "base-general(-typical)" as an alternative for an old notion.

2. Preliminaries on special varieties

Throughout this section, let X be a complex projective manifold and $\mathrm{Div}'(X)$ the set of codimension-one subvarieties of X. An orbifold structure on X is a $\mathbb{Q}$-divisor of the form

$$A = \sum_i (1 - 1/m_i) D_i$$

where $1 \leq m_i \in \mathbb{Q} \cup \{\infty\}$ and $D_i \in \mathrm{Div}'(X)$ for all i. We denote X with its orbifold structure by $X \backslash A$ and we set $K_{X \backslash A} := K_X(A)$ to be the orbifold canonical $\mathbb{Q}$-bundle. We set $m(A \cap D_i) = m(D_i \cap A) = m_i$ and call it the multiplicity of the orbifold $X \backslash A$ (or simply, the orbifold multiplicity) at D_i. We note that the coefficient of D_i in A satisfies

$$0 \leq 1 - \frac{1}{m(A \cap D_i)} \leq 1$$

and it vanishes, respectively equals one, precisely when the corresponding orbifold multiplicity is one, respectively equals ∞. Note that when A is a (reduced) normal crossing divisor, the orbifold $X \backslash A$ is nothing but a log-manifold (X, A) whose birational geometry is dictated precisely by the proper birational geometry of the complement of A in X. More generally, when A_{red} is normal crossing, one can make good geometric sense of the orbifold $X \backslash A$ via the usual branched covering trick (see [16], see also [6] for a variant approach) and we will call such an orbifold smooth.

We now define the Kodaira dimension of a rational map from an orbifold following [16], c.f. also [5]. Let $f : X \dashrightarrow Y$ be a rational map between complex projective manifold and let w be a rational section of K_Y. If f is dominant, then f^*w defines a rational section of Ω_X^m with $m = \dim Y$ and hence determines in the standard way a saturated rank-one subsheaf L of Ω_X^m which is easily seen to be unique in the birational equivalence class of f (it is even unchanged after composing with a dominant map from Y to a variety of the same dimension as Y). We recall that a saturated subsheaf of a locally free sheaf $\mathcal{S}$ is one that is not contained in any larger subsheaf of the same rank and that it is reflexive. It follows that such a subsheaf, if it is rank-one, is locally free (see for example, [18]). Hence, we can even define L without the dominant condition on f by setting $m = \dim f(X)$ and replacing Y by a desingularization of the algebraic image of f in general. Now f gives rise to an orbifold

rational map in the category of orbifold if an orbifold structure A is imposed on X. We denote this orbifold map by f^∂ and the orbifold $X\backslash A$ by X^∂ if A is implicit.

DEFINITION 2.1. *Let $f : X \dashrightarrow Y$ be a rational map giving rise to an invertible sheaf L on X as defined above. Let A be an orbifold structure on X giving rise to an orbifold rational map that we denote by*

$$f^\partial = f|_{X\backslash A} : X\backslash A \dashrightarrow Y.$$

Define the vertical part of A with respect to f by

$$A \cap f = \sum \Big\{ \Big(1 - \frac{1}{m(D \cap A)}\Big) D \ \Big|\ f(D) \neq f(X),\ D \in \mathrm{Div}'(X) \Big\}.$$

We set $L_{f^\partial} = L(A \cap f)$, which is a $\mathbb{Q}$-invertible sheaf, and we define the Kodaira dimension of the orbifold rational map f^∂ by

$$\kappa(f, A) = \kappa(f^\partial) := \kappa(L_{f^\partial}).$$

Recall that a dominant rational map is called almost holomorphic if its general fibers are well defined (i.e., do not intersect with the indeterminancy locus). More specifically, the restriction of the second projection to the exceptional locus of the first projection of the graph of the map is not dominant. Such a map is called an almost holomorphic fibration if the general fibers are connected. Recall also that a fibration is a proper surjective morphism with connected fibers while a fiber space is a dominant morphism whose general fibers are connected. A dominant rational map is called a rational fibration if it becomes a fibration after resolving its indeterminancies.

DEFINITION 2.2. *Notation as above, we call the orbifold rational map f^∂ to be **(base-wise) of general type** (or simply to be **base-general(-typical)**) if*

$$\kappa(f^\partial) = \dim(f) > 0,$$

*where $\dim(f)$ is given by the dimension of the algebraic image of f. We call the orbifold $X\backslash A$ **special** if it admits no base-general orbifold rational map and to be **general-typical** or **of general type** if the identity map restricted to the orbifold is base general. If f is a rational fibration, we say that $f^\partial = f|_{X\backslash A}$ is **base-special** if $X\backslash A$ has no orbifold rational map that is base-general and that factors through f; We will also consider the obvious generalization of this notion to dominant rational maps via Stein factorization. If f is an almost holomorphic fibration, we say that f^∂ is **special** (respectively **general-typical**) if its general fiber endowed with the orbifold structure given by the restriction of A are special orbifolds (repectively orbifolds of general type). It should be clear that orbifold structures under generic restrictions makes sense, see lemma 2.6).*

In the case A is reduced and normal crossing, it is easily seen that these notions are, in the obvious manner, proper birational invariants of the open subset $X \setminus A$ and of the restriction of f to it. Hence, these notions make sense for quasiprojective varieties and mappings from them and we will so understand them in this context.

This notion of being special corresponds to the same "geometric" notion introduced by Campana in [6] in the case A_{red} is normal crossing and to the notion given in section 1 in the case A is reduced and normal crossing by virtue of the following two lemmas respectively, see

[16]. The first of these lemmas is self-evident (with the help of the existence of diagram 2.1 in lemma 2.13 as one convenient but not absolutely necessary shortcut).

LEMMA 2.3. *Let L be a saturated line subsheaf of Ω^i_X and $X \backslash A$ a log-manifold. Then the saturation of L in $\Omega^i(X, \log A)$ is $L(A')$ where A' consists of components D of A whose normal bundles N_D over their smooth loci satisfy $N^*_D \wedge L = 0$ in $\Omega^{i+1}_X|_D$. Hence given a dominant map $f^\partial : X \backslash A \dashrightarrow Y$, L_{f^∂} is the saturation of L_f in $\Omega^r(X, \log A)$, $r = \dim Y$.* ■

LEMMA 2.4 (Bogomolov, Castelnuovo-DeFranchis)**.** *Let L be a saturated line subsheaf of $\Omega^p(X, \log A)$ where A is a normal crossing divisor in X. Then*

(I) *$\kappa(L) \leq p$.*
(II) *If $\kappa(L) = p$, then the Iitaka fibration I_L of L defines an almost holomorphic fibration to a projective base B of dimension p and $I^*_L K_B$ saturates to L in $\Omega^p(X, \log A)$. In particular, $L = L_{I^\partial_L}$.* ■

Now let $f : X \to Y$ be a fibration with X and Y projective and smooth and let A be an orbifold structure on X. Then the induced orbifold fibration $f^\partial = f|_{X\backslash A}$ imposes an orbifold structure on Y as follows. Given $D \in \mathrm{Div}'(Y)$, we may write $f^*D = \sum_i m_i D_i$ for $m_i \in \mathbb{N}$ and $D_i \in \mathrm{Div}'(X)$. Then we define the (minimum) multiplicity of f^∂ over D by

$$m(D, f^\partial) = \min\{\ m_i\, m(D_i \cap A) \mid f(D_i) = D\ \}.$$

DEFINITION 2.5. *With the notation as given above, the $\mathbb{Q}$-divisor on Y*

$$D(f^\partial) = D(f|_{X\backslash A}) = D(f, A) := \sum \Big\{ \Big(1 - \frac{1}{m(D, f^\partial)}\Big) D \ \Big|\ D \in \mathrm{Div}'(Y) \Big\},$$

defines the ***orbifold base*** *$Y \backslash D(f^\partial)$ of $f^\partial = f|_{X\backslash A}$.*

It is immediate that $D(f^\partial)$ is supported on the union of $f((A \cap f)_{red})$ with the divisorial part $\Delta(f)$ of the discriminant locus of f. Note that replacing f by its composition with a birational morphism $r : \tilde{X} \to X$ and imposing the ∞ multiplicity along the exceptional divisor of r while keeping the other orbifold multiplicities the same does not change $D(f^\partial)$. Hence, although the definition of $D(f^\partial)$ would no longer make sense if we allow f to be meromorphic, we can deal with the problem in a consistent way (though not always the best way) by resolving the singularities of f and imposing the ∞ multiplicity along the exceptional divisor of of the resolution. In the case A is reduced, the same can be achieved by imposing only the ∞ multiplicity along the exceptional divisor of r that maps to A, that is, r^∂ gives a proper birational morphism to $X \backslash A$. This is always adopted in the case A is reduced. The following two lemmas (lemma 3.5 and 3.4 of [16]) are essentially immediate consequences of the definition.

LEMMA 2.6. *With the notation as above, let $g : Y \to T$ be a fibration and $h = g \circ f$. Let $i : X_t \hookrightarrow X$, respectively $j : Y_t \hookrightarrow Y$, be the inclusion of the fiber of h, respectively g, above a general point $t \in T$. Then $D(f^\partial)_t := j^* D(f^\partial)$ and $A_t := i^* A$ are orbifold structures on the nonsingular fibers Y_t and X_t respectively and $D(f^\partial)_t = D(f^\partial_t)$, where $f^\partial_t = f_t|_{X_t \backslash A_t}$. That is*

$$D(f|_{X\backslash A})|_{Y_t} = D(f_t|_{X_t\backslash A_t}).$$

Hence $(Y^\partial)_t := Y_t \backslash D(f^\partial)_t$ and f^∂_t make sense and $(Y^\partial)_t = Y_t \backslash D(f^\partial_t) =: (Y_t)^\partial$.

Proof: The lemma follows from the fact that h, respectively g, and its restriction to the divisor $R = (A + f^*\Delta(f))_{red}$ in X, respectively the divisorial part of $f(R)_{red}$, are generically

of maximal rank when resticted to their fibers above t (by Sard's theorem). ■

Hence, the definition of $D(f^\partial)$ is well behaved under generic restrictions.

LEMMA 2.7. *Let f^∂, g, h and A be as above, let $B = D(f^\partial) = D(f, A)$, $g^\partial = g|_{Y\backslash B}$ and $h^\partial = h|_{X\backslash A}$. Then $D(g^\partial) \geq D(h^\partial)$, i.e., $D(g, D(f, A)) \geq D(g \circ f, A)$. If the exceptional part of A with respect to f is reduced or if A and B are reduced and $f^\partial : X\backslash A \to Y\backslash B$ is proper and birational, then equality holds.* ■

The following theorem is the key fact about special orbifolds used to establish our main theorem. It will be used in the next section.

THEOREM 2.8. *Let X^∂ be a (smooth) orbifold, $f^\partial : X^\partial \to T$ a special orbifold fibration and $h^\partial : X^\partial \dashrightarrow Z$ a base-general orbifold rational map. Then $h^\partial = k \circ f^\partial$ for a rational map $k : T \dashrightarrow Z$ and $k^\partial := k|_{T\backslash D(f^\partial)}$ is base-general. In particular, if $T\backslash D(f^\partial)$ is special, then f^∂ is base special and hence X^∂ is special.*

This is Proposition 6.5 of [16], see also [17] and Chapter 8 of [6]. In view of its importance here, we reproduce a proof below adapted to our situation.

Recall that a $\mathbb{Q}$-invertible sheaf is called big if it has maximal Kodaira dimension. We first quote two elementary and well-known lemmas concerning the Kodaira dimension.

LEMMA 2.9 (Kodaira, [15])**.** *Let H and L be invertible sheaves on X with H ample. Then L is big if and only if there is a positive integer m such that* $\dim H^0(L^m H^{-1}) \neq 0$.

LEMMA 2.10 (Easy Addition Law, [12])**.** *Let $f : X \to Y$ be a fibration with general fiber F and L an invertible sheaf on X. Then*

$$\kappa(L) \leq \kappa(L|_F) + \dim(Y).$$

The following is a simplified version of Lemma 5.7 of [16].

LEMMA 2.11. *Consider the following commutative diagram of rational maps between complex projective manifolds*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"] \arrow[dr, "g"] & Z \\ T & Y \arrow[l, "u"] \arrow[u, "w"'] \end{tikzcd}$$

where f is a fibration, g and h are dominant rational maps and u and w are morphisms, necessarily surjective. Let A be an orbifold structure on X. Let i and j be the inclusion of the general fibers $X_t := f^{-1}(t)$ and $Y_t := u^{-1}(t)$ over T. Let $g_t = g \circ i$ and $h_t = h \circ i$. Assume that $w \circ j$ is generically finite so that $L_{g_t} = L_{h_t}$. Then i^*A is an orbifold structure on X_t and we have with $p = \dim Z = \dim Y$, $q = \dim Y_t$ that

$$\kappa(h^\partial) - (p - q) \leq \kappa(i^* L_{h^\partial}) \leq \kappa(L_{h_t^\partial}) =: \kappa(h_t^\partial).$$

In particular, if h^∂ is base-general, then so are h_t^∂ and g_t^∂ if $\dim Y_t > 0$.

Proof: It will be clear from our proof that we may assume WLOG, by taking repeated hyperplane sections of T if necessary, that w is generically finite. So we assume this from the start. Note then that $\dim(T) = p - q$ so that the first inequality above follows from

the easy addition law of Kodaira dimension. To obtain the second inequality and thus the lemma, the easily verified fact that

$$i^*(A \cap h) = (i^*A) \cap (h \circ i) = (i^*A) \cap h_t = (i^*A) \cap g_t$$

allows us to deduce it from an inclusion of i^*L_h in $L_{h_t} = L_{g_t}$ that can be seen as follows. The conormal short exact sequence on X_t

$$0 \to N^*_{X_t} \to \Omega_X|_{X_t} \to \Omega_{X_t} \to 0$$

gives rise to a natural sheaf morphism from $i^*\Omega^p_X = \Omega^p_X|_{X_t}$ to the factor $\Omega^q_{X_t} \otimes \Lambda^{p-q} N^*_{X_t}$ in its quotient filtration. Now, over the Zariski open set U of X_t where g_t and h_t are defined, $i^*g^* = g^*_t$ gives a map from the same short exact sequence on Y_t to that of X_t. So it does the same for the corresponding natural sheaf morphism on Y_t to that of X_t. Thus we obtain a commutative diagram over U:

$$\begin{array}{ccccc}
i^*h^*K_Z & \longrightarrow & \Omega^p_X|_{X_t} & \longrightarrow & \Omega^q_{X_t} \otimes \Lambda^{p-q} N^*_{X_t} \\
\downarrow{\scriptstyle\delta} & & \uparrow & & \uparrow \\
i^*g^*K_Y & \xrightarrow{\sim} & g^*_t(K_Y|_{Y_y}) & \longrightarrow & g^*_t(K_{Y_t} \otimes \det N^*_{Y_t})
\end{array}$$

where δ is induced from the inclusion $w^*K_Z \hookrightarrow K_Y$. As both $\det N^*Y_t$ and $\Lambda^{p-q}N^*_{X_t} = \det N^*_{X_t}$ are trivial invertible sheaves by construction, we see that $g^*_t K_{Y_t}$ has the same image in $\Omega^q_{X_t}$ as that of $i^*h^*K_Z$ over a Zariski open subset of X_t. As the former saturates to L_{g_t} in $\Omega^q_{X_t}$, we see that $i^*L_h \hookrightarrow L_{g_t}$ as required. ■

Proof of Theorem 2.8: Let $g_0 = (f, h) : X \dashrightarrow T \times Z$ and Y_0 its image. Let $r : Y \to Y_0$ be a resolution of singularities of Y_0 and let $g = g_0 \circ r^{-1}$, which is a rational map in general. Let u and w be r composed with the projections of $T \times Z$ to T and Z respectively. Then we are in situation of Lemma 2.11 with h^∂ base-general. We first note that the general fibers of u are connected by construction, being images of the fibers of the special fibration f. As the general fibers of f are special, our lemma implies that Y_t for the general $t \in T$ are points. It follows that u is birational so that Y_0 form the graph of a rational map $k : T \dashrightarrow Z$ and $h = k \circ f$. The theorem now follows directly from the following elementary lemma, which is a simplification of Proposition 3.19 of [16]. ■

LEMMA 2.12. *Let $f : X \to T$ be a fibration, $k : T \dashrightarrow Z$ a rational map and $h = k \circ f$. Let A be an orbifold structure on X inducing the orbifold maps $f^\partial = f|_{X\setminus A}$ and $h^\partial = h|_{X\setminus A}$. Let $B = D(f^\partial)$ be the orbifold structure on T imposed by f^∂ and let $k^\partial = k|_{T\setminus B}$ be the induced orbifold map from T. Then*

$$\kappa(h^\partial) \le \kappa(k^\partial).$$

In particular, if h^∂ is base-general, then so is k^∂.

Proof: We only prove the lemma in the case we are using in the paper where A and B are reduced normal crossing divisors; more specifically, in the case A is reduced normal crossing, f^∂ is a special fibration and and $B = D(f^\partial)$ is just the standard boundary divisor of the compactification of a semi-Abelian variety. In this case, both $X\setminus A$ and $Y\setminus B$ are log-manifold and so L_{f^∂} and L_{k^∂} can be considered respectively as invertible subsheaves of $\Omega(X, \log A)$ and $\Omega(T, \log B)$ by Lemma 2.3. We note that, outside the exceptional divisor $E(f)$ of f, f is a log-morphism (i.e., $f^{-1}(B) \subset A$) and so gives an inclusion of sheaves

$f^*\Omega(T, \log B) \hookrightarrow \Omega(X, \log A)$ there and it is actually a vector bundle inclusion on a Zariski open subset of $f^{-1}(B)$ ([12]). Thus, we have an inclusion

$$f^*L_{k^\partial} \hookrightarrow L_{h^\partial}$$

outside $E(f)$ that is an equality on a Zariski open subset of $f^{-1}(B)$. This equality extends to the open subset outside A ($\supset f^{-1}(B)$) where f is smooth. But by our definition of the multiplicity that gives the orbifold base, this open subset surjects to the complement of a codimension two subet of $T\backslash B$. Hence $H^0(L^l_{h^\partial}) \hookrightarrow H^0(f^*L^l_{k^\partial}) = H^0(L^l_{k^\partial})$ for all positive integer l by the Hartog extension theorem. ■

We remark that in our case at hand, $Y\backslash B$, being a semi-Abelian variety with an equivariant compactification Y (see the next section for the definition and basic facts), has trivial log-cotangent sheaf. Hence $\kappa(L_{k^\partial}) \leq 0$ and $Y\backslash B$ is a special orbifold.

We now address the very important question of when is the base Kodaira dimension of an orbifold fibration equal to the Kodaira dimension of its orbifold base. The question was posed by Campana in [5] for which he gave a partial answer in the case the base has positive Kodaira dimension. We have also given a partial answer in lemma 2.2 and 2.4 of [16] which showed at the same time the equivalence of our approach to that of Campana's. It is this latter that we give below but restricted here for simplicity to the context of log-manifold.

LEMMA 2.13. *Let $f : X \to Y$ be a fibration where X and Y are complex projective manifolds, A a normal crossing divisor on X and $f^\partial = f|_{X\backslash A}$. Then $\kappa(f^\partial) \leq \kappa(Y\backslash D(f^\partial))$. Also, one can find a commutative diagram of morphims between complex projective manifolds*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array} \tag{2.1}$$

with u, v birational and onto such that $E(f' \circ v^{-1}) = \emptyset$ and $A' = v^{-1}(A)$ is normal crossing. Let $f'^\partial = f'|_{X'\backslash A'}$. Then v induces a proper birational morphism $X \setminus A \to X' \setminus A'$ and $\kappa(f^\partial) = \kappa(f'^\partial)$. If m is divisible by the multiplicities of $D(f'^\partial)$, then

$$H^0(X, L^m_{f'^\partial}) = H^0(Y', K^m_{Y'\backslash D(f'^\partial)}) \quad \textit{and} \quad \kappa(f'^\partial) = \kappa(Y'\backslash D(f'^\partial)).$$

Proof: The first statement follows from lemma 2.12 by letting k be the identity map there.

The construction of a birationally equivalent fibration as given by the commutative diagram with the above property is achieved by resolving the singularities of the flattening of f, which exists by [23, 8], and in such way that the inverse image of A is normal crossing, which is possible by [10]. As u is birational, $v^*L_{f^\partial} \hookrightarrow L_{f'|_{X'\backslash A'}}$ by lemma 2.3 and hence $\kappa(L_{f^\partial}) \leq \kappa(L_{f'|_{X'\backslash A'}})$. The reverse inequality follows from lemma 2.12.

For the last statement, we have (with $r = \dim Y'$) as before the inclusion

$$f'^* K_{Y'}(D(f'^\partial))^m \hookrightarrow L^m_{f'^\partial} \left(\hookrightarrow (\Omega^r_{X'}(\log A'))^{\otimes m} \right) \tag{2.2}$$

outside $O \cup E(f')$ where O is a subset of X' of codimension two or higher contained above the discriminant $\Delta(f')$ of f' and m is a positive multiple of all relevant multiplicities. Moreover, this inclusion is an isomorphism on an open subset of X' that surjects to the complement of a subset of Y' of codimension two or higher. Hence $H^0(Y', K_{Y'}(D(f'^\partial))^m) \hookrightarrow H^0(X, L^m_{f^\partial}) =$

$H^0(X', L^m_{f'^\partial})$ by the Hartog extension theorem applied to X and the reverse inclusion by the Hartog extension theorem applied to Y'. ■

We give below generalizations to the relative setting of lemma 2.13 and theorem 2.8. They are used to extend our main theorem but are otherwise not needed for its proof.

LEMMA 2.14. *With the setup as in lemma 2.13 and with all elements of the commutative diagram (2.1) as given there, let $g' : Y' \dashrightarrow Z'$ be a dominant rational map. Then*

$$\kappa(g', D(f'|_{X'\setminus A'})) = \kappa(g' \circ f', A').$$

Proof: The proof is the same as that of lemma 2.13, replacing $K_{Y'}(D(f'^\partial))$ by $L_{g'|_{Y'\setminus D(f'^\partial)}}$. ■

PROPOSITION 2.15. *Let the setup be as in lemma 2.6, i.e., $f : X \to Y$ and $g : Y \to T$ are fibrations with orbifold structures A on X and $D(f, A)$ on Y and t a general point on T. Assume A is a normal crossing divisor. If f^∂ and $Y_t^\partial = Y_t \setminus D(f|_{X_t^\partial})$ are special, then so is $X_t^\partial := X_t \setminus A_t$. If $T \setminus D(g \circ f, A)$ is special, then f^∂ is base-special if so is f_t^∂.*

Proof: The last statement follows by noting that f^∂ is base-special if so is f'^∂, in the notation of lemma 2.13, and f'^∂ is base-special by lemma 2.14 and so theorem 2.8 applies. ■

3. Structure of the quasi-Albanese map

Let X be a complex quasi-projective manifold, T a semi-Abelian variety and $u : X \to T$ an algebraic morphism. Let $\bar{T}$ be a smooth equivariant compactification of T, i.e., $\bar{T}$ is smooth and admits an algebraic action by T – an example being the compactification of T in the exact sequence (1.1) via the compactification $\mathbb{C}^k \subset (\mathbb{P}^1)^k$. Then one can observe, see [19], that $\bar{T} \setminus T$ is a normal crossing divisor and that $\bar{\Omega}_T$ is a trivial bundle over $\bar{T}$ (via simultaneous equivariant resolution of singularities for example). By the resolution of singularity theorem (see [4, 10]), there is a compactification $\bar{X}$ of X with normal crossing boundary divisor A such that u extends to a morphism $\bar{u} : \bar{X} \to \bar{T}$.

DEFINITION 3.1. *We call $\bar{u}$ as above a* ***natural compactification*** *of u. We will set $\bar{u}^\partial = \bar{u}|_{\bar{X}\setminus A}$ and, in the case u is a fiber space (i.e., $\bar{u}$ is a fibration), we set $D(u) := D(\bar{u}^\partial)|_X$.*

We note that $D(u)$ is a $\mathbb{Q}$-divisor on X that is independent of the natural compactification $\bar{u}$ of u chosen since two such compactifications are always dominated by a third such compactification. By the same token, the notions of being special, being general-typical, being base-special and being base-general(-typical) are well-defined for u (independent of the natural compactifications).

DEFINITION 3.2 ([21])**.** *Let D be an algebraic subset or a $\mathbb{Q}$-divisor in T. We define $\mathrm{St}(D)$ to be the identity component of $\{a \in T : a + D = D\}$ which is easily verified to be a subgroup, even a semi-Abelian subvariety. Given a compactification $\bar{T}$ of T, we define $\bar{D}$ to be the Zariski closure of D in $\bar{T}$.*

The purpose of this section is to establish the following proposition but restricted to the situation we are in.

PROPOSITION 3.3. *Let X be a complex quasi-projective manifold with (quasi-)Albanese map $f : X \to T_0 = \mathrm{Alb}(X)$. If f is not base special, then there is a proper semi-abelian*

subvariety T' of T_0 such that if $g : T_0 \to T = T_0/T'$ is the quotient map, then the orbifold base of $h = g \circ f$ is of general type and of positive dimension; More specifically, if $\bar{e} : \bar{Z} \to \bar{T}$ is the finite map factor in the Stein factorization of $\bar{h}$, then $\dim \bar{Z} > 0$ and either $Z = \bar{e}^{-1}(T)$ is of general type or h is a fiber space (i.e., $\bar{h}$ a fibration) and $K_{\bar{T}}(D(\bar{h}, A)) = \mathcal{O}(\bar{D})$ is big for one and hence for all natural compactification $\bar{h} : \bar{X} \to \bar{T}$ of h with $A = \bar{X} \backslash X$ and $D = D(h)$.

Suppose that f is a fiber space and let $h_0 = f$. Consider the following inductive definition. With h_{i-1} and T_{i-1} defined, we define $D_{i-1} = D(h_{i-1})$, $T''_i = \mathrm{St}(D_{i-1})$, $T_i = T_{i-1}/T''_i$ and $h_i = \gamma_i \circ h_{i-1}$ with $\gamma_i : T_{i-1} \to T_i$. Then this process terminates at the l-th stage for l such that $\mathrm{St}(D_l) = \{0\}$. Setting $h = h_l$ $D = D(h)_{red}$ and $T = T_l$, we find two possibilities:

(i) $\dim T > 0$ *and $\mathcal{O}(\bar{D})$ on $\bar{T}$ is big for any equivariant compactification $\bar{T}$ of T.*
(ii) *T reduces to a point and f is base-special.*

In order to prove this proposition, we first recall the structure theorem of Kawamata [13], Kawamata-Viehweg [14] and Ueno [24] concerning the Albanese map:

PROPOSITION 3.4. *Let Z be a normal quasiprojective variety with a finite morphism to a semi-Abelian variety T. Then there is a finite extension $\mathrm{St}(Z)$ of an abelian subvariety of T whose natural action on T lifts to Z and $Z/\mathrm{St}(Z)$ is of general type of dimension $\bar{\kappa}(Z) = \dim Z - \dim \mathrm{St}(Z) \geq 0$. In particular, f is an étale covering map over a translate of a semi-Abelian subvariety of T if and only if $\bar{\kappa}(Z) = 0$ if and only if Z is semi-Abelian.* ■

Since the image of the Albanese map $f = \alpha_X$ of X generates $\mathrm{Alb}(X)$ (whose fact is equivalent to the universal property of the Albanese map), applying the above proposition to the finite map factor of the Stein factorization of f yields directly proposition 3.3 in the case f is not a fiber space. In the case f is a fiber space, the first paragraph of the proposition follows directly from the other part of the proposition. Hence, it remains to establish the two cases (i) and (ii) of the proposition to end the proof of the proposition. We need the following lemma, well-known in the case of Abelian varieties but generalized to the semi-Abelian case by Proposition 3.9 of [21].

LEMMA 3.5. *Let D be an effective $\mathbb{Q}$-divisor in a semi-Abelian variety T, $\bar{T}$ an equivariant compactification of T and $\bar{D}$ the Zariski closure of D in $\bar{T}$. The the conditions $\mathrm{St}(D) = \{0\}$, $\mathcal{O}(\bar{D})$ being big, and $\mathcal{O}(\bar{D}_{red})$ being big are equivalent.*

We remark that for an effective $\mathbb{Q}$-divisor D, there exists $m > 0$ such that $\frac{1}{m} D_{red} \leq D \leq m D_{red}$ and $\mathrm{St}(D) = \mathrm{St}(D_{red})$. Hence, the lemma follows trivially from the weaker assumption that D is a an effective (even a reduced) divisor. We remark also that one can give a direct proof of the above lemma using the original arguments of the above structure theorem, or by applying the above theorem to suitable ramified covers of T.

Proof of proposition 3.3: The claim for case (i) is just lemma 3.5. Let $g_i = \gamma_i \circ \cdots \circ \gamma_1$. To prove (ii), it suffices to prove for general $t_i \in T_i$ that $f_{t_i} = f|_{h_i^{-1}(t_i)}$ is base special for all i. But this follows by induction as follows. It is clear for $i = 0$. Assume that it is true for $f_{t_{i-1}}$. The orbifold base of $h_{i-1}|_{t_i} = g_{i-1} \circ f_{t_i}$, being the restriction of that of h_{i-1} to $\gamma_i^{-1}(t_i)$, is simply $\gamma_i^{-1}(t_i)$ by lemma 2.6, which is a general translate of the semi-Abelian subvariety T''_i and is thus special. As $f_{t_i}|_{t_{i-1}} = f_{t_{i-1}}$ is base-special by the induction hypothesis, lemma 2.15 shows that f_{t_i} is base-special as required. ■

4. Implication of Zariski dense entire curves

We first recall some relevant definitions and facts from Nevanlinna theory. We will follow section 2 of [21]. Let T be a complex manifold, ω a real smooth $(1,1)$-form and $\gamma : \mathbb{C} \to T$ a holomorphic map. Then the order function of γ with respect to ω is defined by

$$T_\gamma(r;\omega) = \int_1^r \frac{dt}{t} \int_{|z|<t} f^*\omega \qquad (r>1). \tag{4.1}$$

If T is Kähler and ω, ω' are d-closed real (1,1)-forms in the same cohomology class $[\omega]$, then

$$T_\gamma(r;\omega) = T_\gamma(r;\omega') + O(1).$$

Hence we may set, up to $O(1)$-terms,

$$T_\gamma(r;[\omega]) = T_\gamma(r;\omega). \tag{4.2}$$

Let $L \to T$ be a hermitian line bundle. As its Chern class is a real $(1,1)$-class, we may set

$$T_\gamma(r;L) = T_\gamma(r;c_1(L)).$$

We will denote by $\mathcal{O}_T(D)$ the line bundle determined by a divisor D on T via a standard abuse of notation and set $T_\gamma(r;D) = T_\gamma(r;\mathcal{O}_T(D))$. By using a standard Weil function for to a subscheme W of T (see section 2 of [21]), we can define $T_\gamma(r;W)$ similarly, which we will also denote by $T_\gamma(r;\mathcal{I}_W)$ where $\mathcal{I}_W$ is the ideal subsheaf of $\mathcal{O}_T$ defining W. This is because, ideal sheaves pulls back to the same on $\mathbb{C}$ which are then ideal sheaves defining effective divisors on $\mathbb{C}$. Henceforth, we will identify ideal sheaves with their effective divisors on $\mathbb{C}$.

Let $E = \sum_{z\in\mathbb{C}}(\mathrm{ord}_z E)z$ be an effective divisor on $\mathbb{C}$, $S \subset \mathbb{C}$ and $l \in \mathbb{N} \cup \{\infty\}$. Then the above sum is a sum over a discrete subset of $\mathbb{C}$. Hence the sum is finite when restricted to the disk $\mathbb{D}_t$ of radius $t>0$ and so

$$n(t;E) = \deg_{\mathbb{D}_t} E := \sum_{z\in\mathbb{D}_t} \mathrm{ord}_z E$$

makes sense. We define the restriction of E to S truncated to order l by

$$E_{S,l} = \sum_{z\in S} \min(\mathrm{ord}_z E, l) z$$

and set $n_l(t;E,S) = n(t;E_{S,l})$. Then the counting functions of E with, respectively without, truncation to order l are given by

$$N_l(r;E,S) = \int_1^r \frac{n_l(t;E,S)}{t} dt\ , \quad N_l(r;E) = N_l(r;E,\mathbb{C}),$$

respectively $N(r;E) = N_\infty(r;E)$.

A well-known consequence of these definitions via the classical Jensen formula is the First Main Theorem:

$$N(r;\gamma^*\mathcal{I}_W) \le T_\gamma(r;W) + O(1), \tag{4.3}$$

where W is a subscheme of T and $\gamma^*\mathcal{I}_W$, as an ideal subsheaf of $\mathcal{O}_\mathbb{C}$, is identified with a divisor on $\mathbb{C}$. By the linearity of n_l and hence of N_l with respect to the third variable, if T is a disjoint union of U and V, then

$$N_l(r;\gamma^*\mathcal{I}_W) = N_l(r;\gamma^*\mathcal{I}_W, \gamma^{-1}(U)) + N_l(r;\gamma^*\mathcal{I}, \gamma^{-1}(V)).$$

Also if $Z \subset E$ is an inclusion of reduced algebraic subsets of T and $Z = \operatorname{supp} \mathcal{I}$, then

$$N(r;\gamma^*\mathcal{I}_Z) \geq N(r;(\gamma^*\mathcal{I}_Z)_{red}) = N_1(r;\gamma^*\mathcal{I}_Z) = N_1(r;\gamma^*\mathcal{I}_E,\gamma^{-1}(Z)). \tag{4.4}$$

By definition, we also have for D a strictly effective divisor on T that $T_\gamma(r;D) \geq 0$ if the image of γ is not in D_{red}. This fact along with Kodaira's lemma (2.9) and linearity of T_γ with respect to to the second variable yields easily the following (which is lemma 2.3 of NWY2):

LEMMA 4.1. *Suppose that T is a complex projective manifold with a big divisor A. Then*

$$T_\gamma(r) = O(T_\gamma(r;A)).$$

Here by convention $T_\gamma(r) := T_\gamma(r;H)$ for an ample divisor H on T.

We will need the following two theorems of Noguchi-Winkelmann-Yamanoi.

THEOREM 4.2 ([22])**.** *Let X be a normal complex quasi-projective variety admitting a Zariski dense entire holomorphic curve. Let $f : X \to T$ be a finite morphism to a semi-Abelian variety. Then f is an étale covering morphism.*

THEOREM 4.3 ([21])**.** *Let T be a semi-abelian variety and $\gamma : \mathbb{C} \to T$ a holomorphic map with Zariski dense image. Let E be a divisor on T and $\bar{E}$ be its Zariski closure in a equivariant compactification of T. Let $\mathcal{I}$ be an ideal subsheaf of $\mathcal{O}_T$ such that $\mathcal{O}_T/\mathcal{I}$ is supported on a codimension-two subvariety of T. Then we have :*

$$N(r;\gamma^*\mathcal{I}) \leq \epsilon T_\gamma(r)||_\epsilon \text{ for all } \epsilon > 0 \tag{4.5}$$

$$T_\gamma(r;\bar{E}) \leq N_1(r;\gamma^*E) + \epsilon T_\gamma(r;\bar{E})||_\epsilon \text{ for all } \epsilon > 0 \tag{4.6}$$

Here "$||_\epsilon$" stands for the inequality to hold for every $r > 1$ outside a Borel set of finite Lebesgue measure that depend on ϵ.

COROLLARY 4.4. *Let X be a smooth complex quasi-projective variety admitting a Zariski dense entire holomorphic curve. Then the (quasi-)Albanese map of X is a fiber space.*

Proof of corollary: This is an easy argument applying theorem 4.2 to the Stein factorization of a natural compactification of f. ■

Our main theorem in this paper is a direct consequence of the following proposition.

PROPOSITION 4.5. *Let X be a smooth complex quasi-projective variety admitting a Zariski dense entire holomorphic curve. Then its (quasi-)Albanese map $\alpha : X \to T_0 =$ $\mathrm{Alb}(X)$ is a fiber space and α is base special.*

Proof: The first part of the proposition is just corollary 4.4. We may thus assume that α is a fiber space but not base special. The proposition is proved once we reach a contradiction with the existence of an entire map $\gamma_0 : \mathbb{C} \to X$ with Zariski dense image in T_0. By proposition 3.3(i), there is a quotient morphism $g : T_0 \to T$ of semi-Abelian varieties such that the fiber space $h = g \circ f : X \to T$ induces an orbifold base of general type with $\dim T > 0$, that is, given any equivariant compactification $\bar{T}$ of T, $\mathcal{O}(\bar{D})$ is big where $D = D(h)_{red}$. Let $\gamma = h \circ \gamma_0$. As γ has Zariski dense image in T, theorem 4.3 gives

$$T_\gamma(r;\bar{D}) \leq N_1(r;\gamma^*D) + \epsilon T_\gamma(r;\bar{D})||_\epsilon \text{ for all } \epsilon > 0 \tag{4.7}$$

From the definition of $D(h)$, we see that outside the exceptional locus E of h, the effective divisor $D_0 = h^*(D)$ is nowhere-reduced along $h^{-1}(D)$. Since $Z = h(E)$ is of co-dimension two or higher, we have from theorem 4.3 that

$$N(r;\gamma^* Z) \leq \epsilon T_\gamma(r)||_\epsilon \text{ for all } \epsilon > 0. \tag{4.8}$$

As $\gamma^*(D) = \gamma_0^*(D_0)$ is then an effective divisor that is nowhere-reduced outside $\gamma^{-1}(Z)$, we have by (4.4) that

$$N(r;\gamma^*(D)) \geq 2N_1(r;\gamma^* D, \gamma^{-1}(T \setminus Z)) \geq 2N_1(r;\gamma^*(D)) - 2N(r;\gamma^*\mathcal{I}_Z). \tag{4.9}$$

Also, since $\bar{D}$ is big, we have by lemma 4.1, that $T_\gamma(r) = O(T_\gamma(r;\bar{D})$. That is, there exists a constant $C > 0$ (depending on the ample divisor used to define $T_\gamma(r)$) such that

$$T_\gamma(r) \leq CT_\gamma(r;\bar{D}).$$

Couple all these with the First Main Theorem, equation (4.3), gives

$$\begin{aligned} T_\gamma(r;\bar{D}) + O(1) &\geq N(r;\gamma^* D) \geq 2N_1(r;\gamma^* D) - 2N(r;\gamma^* Z) \\ &\geq (1-\epsilon)2T_\gamma(r;\bar{D}) - 2\epsilon C T_\gamma(r;\bar{D})||_\epsilon \\ &\geq 2(1-\epsilon-\epsilon C)T_\gamma(r;\bar{D})||_\epsilon, \end{aligned}$$

valid for all $\epsilon > 0$. This gives a contradiction for $\epsilon > 0$ sufficiently small. ■

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